Section I

10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1 – 10

1. The amplitude of the curve $y = \frac{1}{2}\sin(3x + \frac{\pi}{6})$ is **MA12.5**

A. $\frac{1}{6}$

B. $\frac{\pi}{6}$

C. $\frac{1}{3}$

D. $\frac{1}{2}$

2. What is the equation of the tangent to the curve $y = \sin x$ at the origin? **MA12.6**

A. $y = -x$

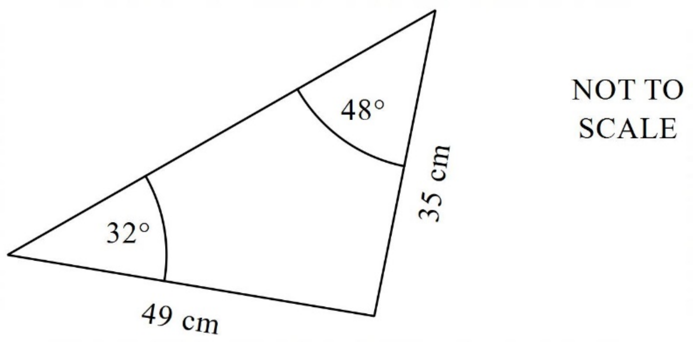
B. $y = \cos x$

C. $y = \sin x$

D. $y = x$

3. Calculate the area of the triangle below. **MA11.4**

(Answer to the nearest cm^2.)



A. $422\ cm^2$

B. $637\ cm^2$

C. $844\ cm^2$

D. $858\ cm^2$

4. Given that $\log_a b = 3.75$ and $\log_a c = 0.25$. **MA11.6**

Find $\log_a (bc)^2$.

A. 0.9375

B. 3.5

C. 8

D. 0.879

5. Find the exact value of $\int_0^4 \sqrt{16 - x^2}\, dx$. **MA12.7**

A. 4 $units^2$

B. 16π $units^2$

C. 2 $units^2$

D. 4π $units^2$

6. The point $P(8, -3)$ lies on the graph of $y = f(x)$. **MA12.1**

Find the coordinates of the image of P if the function is transformed to :

$y = -2f(x+7) + 5$.

A. (1, 11)

B. (1, –1)

C. (15, 11)

D. (15, –1)

7. Which interval gives the range of the function **MA12.1**

$y = 2\sqrt{25 - x^2}$?

A. [- 5 , 5]

B. [0 , 10]

C. [-10 , 10]

D. [0 , 5]

8. Simplify $\sec^2\theta - \tan^2\theta$. **MA11.4**

A. 1

B. $\sin^2\theta$

C. $\cot^2\theta$

D. 2

9. The third term of an arithmetic series is 32 and the sixth term is 17. **MA12.4**

Find the sum of the first ten terms.

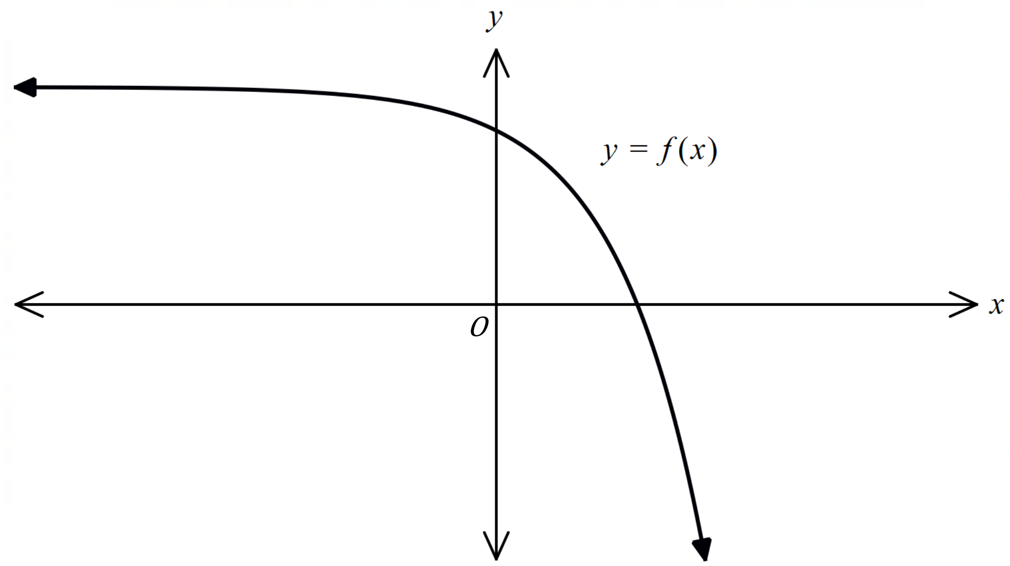
A. - 5

B. 42

C. 49

D. 195

10. The graph of the relation $y = f(x)$ is shown below: **MA12.3**



Which statement is correct?

A. $f\,'(x) > 0$ and $f\,''(x) < 0$

B. $f\,'(x) < 0$ and $f\,''(x) < 0$

C. $f\,'(x) > 0$ and $f\,''(x) > 0$

D. $f\,'(x) < 0$ and $f\,''(x) > 0$

END OF SECTION I

Section II: Questions 11-33

Instructions
- Answer the questions in the spaces provided. Sufficient spaces are provided for typical responses.
- Your responses should include relevant mathematical reasoning and/or calculations.
- Extra writing space is provided at the back of the booklet. If you use this space, clearly indicate which question you are answering.

Question 11 (7 marks)

(a) Express n in terms of x and y, given $(\sqrt{2})^n = \frac{4^x}{32^y}$ **1**

MA12.1

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(b) One of the two solutions to the equation below is $x = 1$ **2**

$2^{2x+y} - 2^{x+y} = 9(2^x) - 2$. Find the value of y. **MA12.1**

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(c) What is the other value of x for the equation. **2**

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(d) Find the values of a and b such that

$$\sum_{n=1}^{3} \log_2 2n = a + \log_2 b$$

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MA12.4

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Question 12 (2 marks) **MA12.1**

The damage caused by a moving car when it hits an object is called the 'collision impact' and is proportional to the square of the speed of the car.

i) Write and equation to describe the relationship between collision impact (C) and the speed of the car (V) for some constant (K). **1**

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ii) What happens to the collision impact when the speed of a car is reduced to one third? **1**

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Question 13 (4 marks)

Desmond tosses a pair of dice. The sum on the uppermost faces are recorded.

(a) i. What is the probability that the sum is 9? **1**

MA11.7

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ii. Find the probability that he will throw a sum of 9 before he throws a sum of 7. **2**

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(b)

If X is a random variable such that $P(X \geq 3) = m$ and $P(X \leq 9) = n$. Show that $P(X < 3|X > 9)$ is $\frac{m-1}{n-1}$ **1**

MA11.7

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Question 14 (4 marks)

Differentiate the following with respect to x:

(a) $\dfrac{\cos\ 3x}{x^2}$. **2**

MA12.6

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(b) $\ln\ \sqrt{3 - x}$. **2**

MA12.6

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Question 15 (3 marks)

(a) Find $\int (3x - 4)^8 \, dx$. **1**

MA12.7

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(b) Find $\int \frac{4 \sin\left(\frac{5x}{3}\right)}{7} \, dx$. **2**

MA12.7

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Question 16(4 marks) **MA11.3**

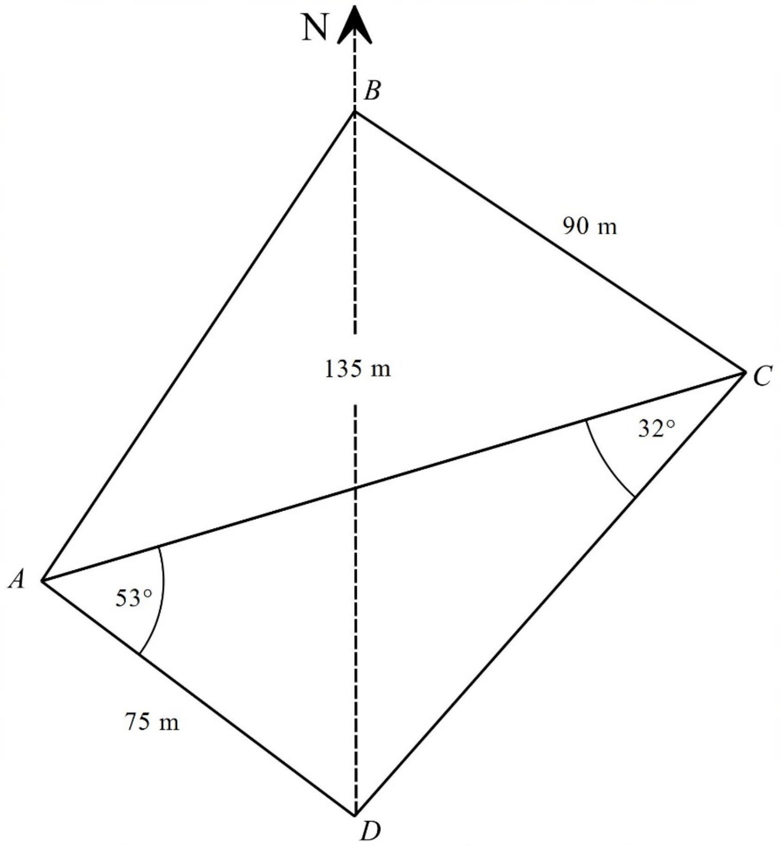
A block of land is in the shape of the quadrilateral *ABCD* shown below. The diagonal *AC* has been drawn.

The following measurements have been taken:

AD = 75 m, *BC* = 90 m.

D is 135 m due south of *B*.

$\angle CAD = 53°$ and $\angle ACD = 32°$.



(a) Find the length of *CD*. (Answer correct to the nearest metre.) **2**

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(b) Find the size of $\angle CBD$ and hence give the bearing of *C* from *B*. **2**
(Give your answer correct to the nearest degree.)

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Question 17 (7 marks)

(a) Use the differentiation by first principles to verify that if, the gradient function of $y = 3x^2 - x$ is $y' = 6x - 1$. **2**

MA12.3

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(b) The line $y = mx + c$ is a normal to the curve $y = x^3 - 3x + 1$ at the point (-2,-1). **2**

Find the values of *m* and *c*. **MA12.3**

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(c) If $a = e^x$, simplify $\log_e a^2$ **1**

MA11.6

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(d) Solve: **2**

MA11.6

$$\log_e x - \frac{3}{\log_e x} = 2$$

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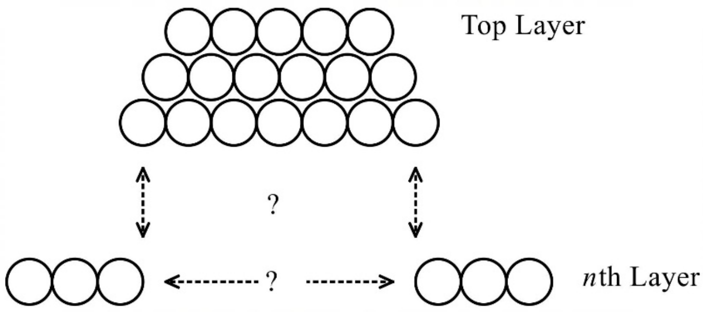
Question 18 (4 marks) **MA12.4**

Lachlan works in a Moolworths store and he is making a stack of oranges against a sloping display panel.

The oranges are stacked in layers, as shown, where each layer contains one orange less than the layer below it.



When he has finished, there are five oranges in the top layer, six in the next and so on.

There are n layers altogether.

(a) Show there are $\frac{1}{2}n(n+9)$ oranges in the stack. **2**

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(b) If Lachlan has 300 oranges to create his display, how many full rows can he create, if the top row still contains five oranges? **2**

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Question 19 (3 marks) **MA11.7**

The random variable ***X*** has this probability distribution.

X	0	1	2	3	4
$\Pr(X = x)$	0.1	0.2	0.4	0.2	0.1

Find

i. $P(X > 1|X \leq 3)$ **1**

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ii. $P(X)$, the variance of X **2**

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Question 20 (3 marks) **MA11.4**

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(a) Express $5cot^2x - 2cosecx + 2$ in terms of $cosecx$

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(b) Hence, solve the equation $5cot^2x - 2cosecx + 2 = 0 \; for \; 0 \leq x \leq 2\pi$.

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Question 21 (3 marks)

(a) Find $\frac{d}{dx}(\log_e(cosx))$ **1**

MA12.7

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(b) Hence, show that the area enclosed by the curve $y = tanx$, the x- axis and the ordinate $x = \frac{\pi}{3}$ is $\boldsymbol{ln2.}$

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MA12.7

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Question 22 (4 marks)

An infinite geometric series is given by $1 - tan^2x + \ tan^4x - tan^6x + \cdots$. for $0 \leq x \leq 2\pi$

(a) Find the limiting sum of the series in simplest form **2** **MA12.4**

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(b) For what values of *x* is the limiting sum is equal $\frac{1}{4}$? **2** **MA12.4**

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Question 23 (3 marks)

(a) Complete the table of values for $y = \sqrt{1 - x^2}$. **1**

Answer to 3 significant figures where required. **MA12.7**

x	0	0.125	0.25	0.375	0.5
y			0.968		0.866

(b) By using the Trapezoidal rule with 4 sub intervals, estimate the integral **2**

$$\int_0^{\frac{1}{2}} \sqrt{1-x^2}\, dx \, .$$

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Question 24 (3 marks) **MA11.7**

The manager of a team notices that the team has a probability of $\frac{2}{3}$ of winning the game if it is raining and if it is dry, the probability of the team winning is $\frac{1}{5}$. The probability that it will rain on a day when they play is $\frac{1}{4}$.

(a) Find the probability that they will not win. **1**

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(b) Given that the team has won the game, calculate the probability that it rained on the day of the match. **2**

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Question 25 (7 marks) **MA12.3**

Consider the curve $y = \frac{1}{4}x^4 - x^3$.

(a) Find any stationary points and determine their nature. **3**

(b) Find any points of inflection. **1**

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(c)

Sketch the curve for $-1.5 \leq x \leq 4.5$, showing all x- intercepts.

(d) For what values of x is the curve concave down? **1**

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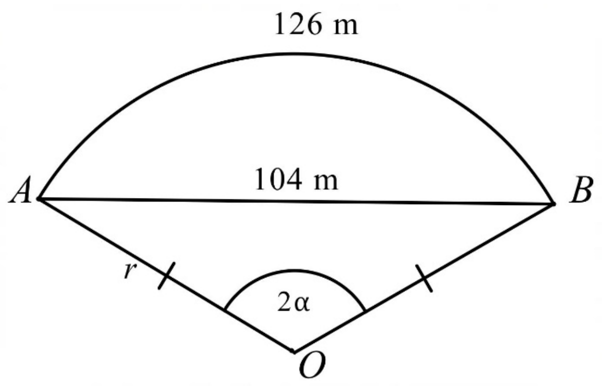
Question 26 (4 marks) **MA11.3**

A sector of a circle, centre *O,* is shown below.

The points *A* and *B* lie on the circle, such that the length of the chord *AB* is 104 metres, and the length of the arc *AB* is 126 metres.

The radius is *r* metres and the angle subtended at the centre by the arc is 2α radians.



(a) Show that $\sin\alpha = \frac{52\alpha}{63}$. **2**

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(b) If $\alpha = \frac{\pi}{3}$, find the exact radius of the circle. **2**

Express your answer with a rational denominator.

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Question 27 (3 marks) **MA12.3**

If $y = \frac{\log_e x}{x}$

(a) Find $\frac{dy}{dx}$ **1**

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(b) Hence show that $\int_e^{e^2} \frac{1-\log_e x}{x \log_e x}\, dx = \log_e 2 - 1$ **2**

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Question 28 (4 marks)

The length of daylight, *L(t)*, is defined as the number of hours from sunrise to sunset, and can be modelled by the equation $L(t) = 12 + \cos\left(\frac{2\pi t}{366}\right)$ where *t* is the number of days after 21 December 2015, for $0 \leq t \leq 366$. **MA12.5**

(a) Find the length of daylight on 21 December 2015. **1**

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(b) What is the shortest length of daylight? **1**

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(c) What are the two values of *t* for which the length of daylight is 12? **2**

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Question 29 (3 marks)

Given that the function $f(x)$ has a derivative $y' = 4e^{4x} + 3$ and the equation of the tangent to this curve is $y = 7x + 2$. Find the exact value of $f(3)$. **3**

MA12.6

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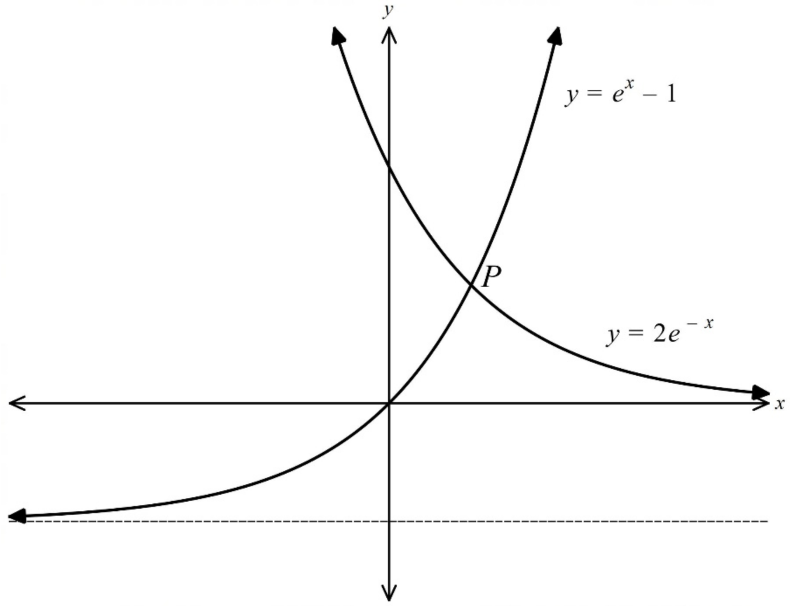
Question 30 (4 marks)

Two curves $y = 2e^{-x}$ and $y = e^{x} - 1$ intersect at a point P.



(a) Show, algebraically, that the coordinates of P are $(\ln 2,\ 1)$. **2**

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(b) Find the area bounded by the two curves and the y-axis. **2**

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Question 31 (2 marks) **MA12.4**

Alex buys a tractor under a buy-back scheme. This scheme gives Alex the right to sell the tractor back to the dealer.

The recurrence relation below can be used to calculate the price Alex sells the tractor back to the dealer (P_n), after n years.

$$\boldsymbol{P_{0=56000}}, \quad \boldsymbol{P_{n=}\, P_{n-1} - 7000}$$

(a) Write the general rule to find the value of $\boldsymbol{p_n}$ in terms of n. **1**

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(b) After how many years will the dealer offer to buy back Alex's tractor at half of its original value. **1**

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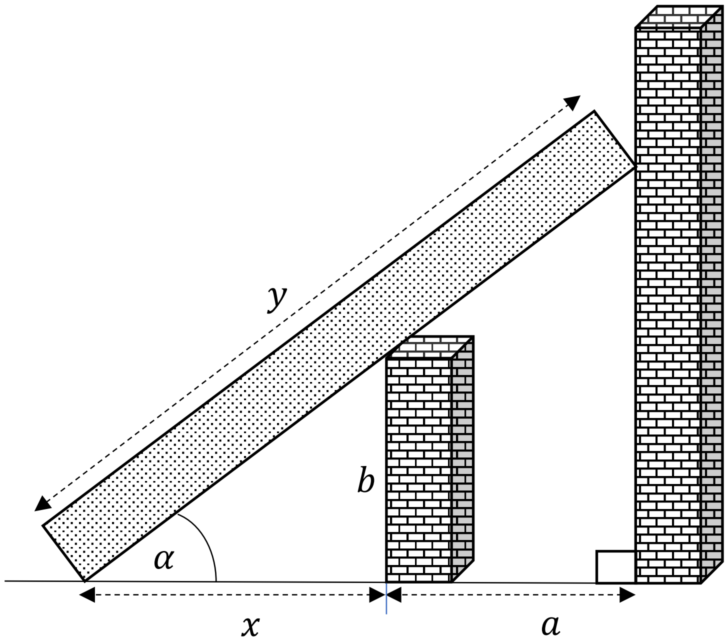
Question 32 (6 marks) **MA12.3**

A vertical wall in danger of collapse is to be braced by a beam, which must pass over a second lower wall b metres high and located a metres from the first wall. Let the length of the beam be y metres, the angle the beam makes with the horizontal be α and x is the distance from the foot of the beam to the smaller wall.



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i. Show that $y = a\sec\alpha + b\operatorname{cosec}\alpha$

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ii. By finding the stationary points on the curve $y = asec\alpha + bcosec\alpha$,

Prove that $\tan\alpha = \sqrt[3]{\frac{b}{a}}$ **2**

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iii. Hence show that the shortest beam that can be used is given by **2**

$$y = a\sqrt{1 + \left(\frac{b}{a}\right)^{\frac{2}{3}}} + b\sqrt{1 + \left(\frac{a}{b}\right)^{\frac{2}{3}}}$$

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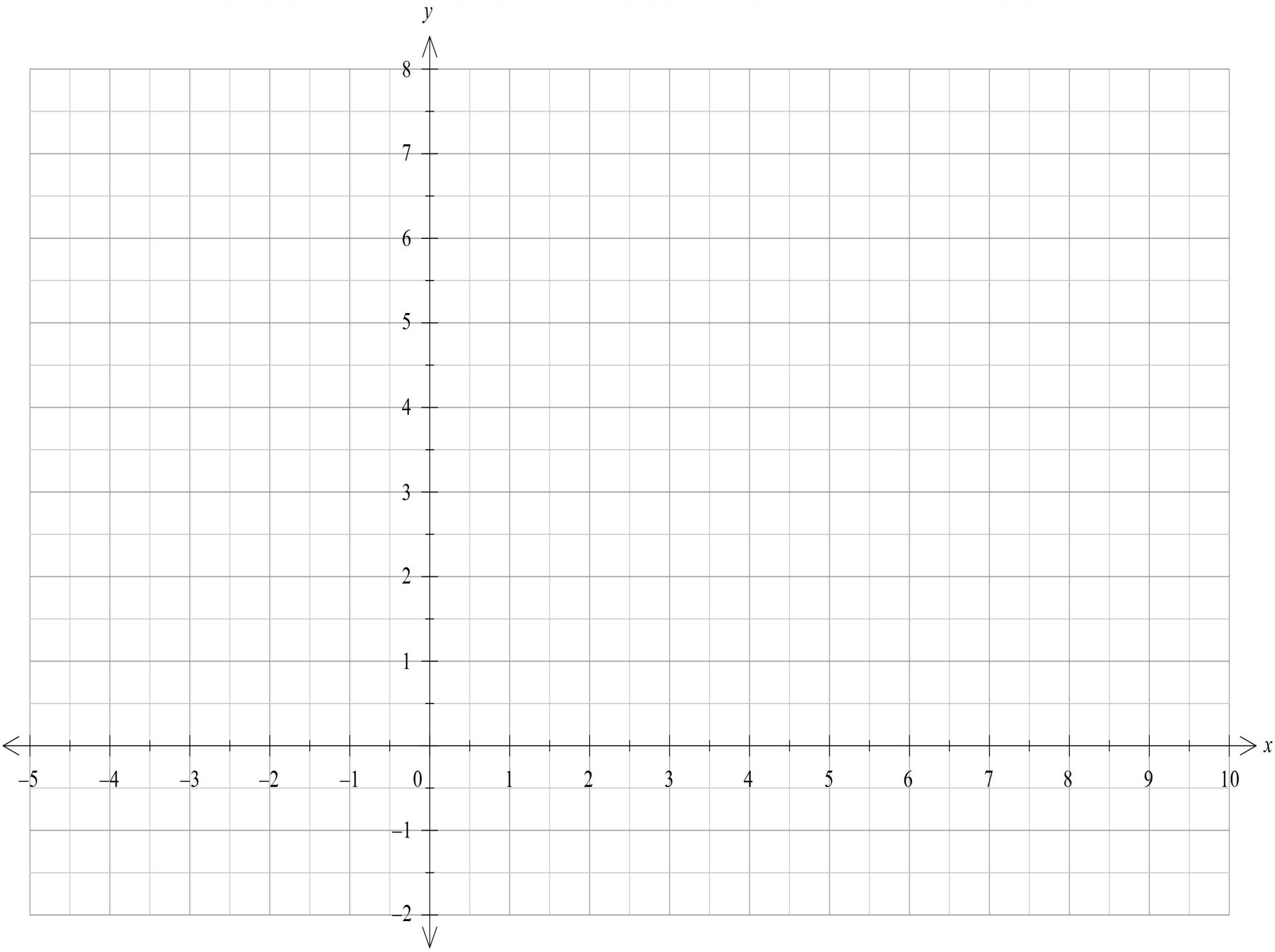
Question 33 (3 marks) **MA12.1**

On the axes provided below, draw a sketch of $y = f(x) = \sqrt{x}$ and use this to draw a sketch of $y = g(x) = 2\sqrt{x+4} - 1$ on the same set of axes. **3**



End of Paper

Section II Extra writing space

If you use this space, clearly indicate which question you are answering.

Section II Extra writing space

If you use this space, clearly indicate which question you are answering.

2021 Trial HSC Examination

Mathematics Advanced
Mathematics Extension 1
Mathematics Extension 2

REFERENCE SHEET

Measurement

Length

$$l = \frac{\theta}{360} \times 2\pi r$$

Area

$$A = \frac{\theta}{360} \times \pi r^2$$

$$A = \frac{h}{2}(a+b)$$

Surface area

$$A = 2\pi r^2 + 2\pi rh$$

$$A = 4\pi r^2$$

Volume

$$V = \frac{1}{3}Ah$$

$$V = \frac{4}{3}\pi r^3$$

Functions

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

For $ax^3 + bx^2 + cx + d = 0$:

$$\alpha + \beta + \gamma = -\frac{b}{a}$$

$$\alpha\beta + \alpha\gamma + \beta\gamma = \frac{c}{a}$$

and $\alpha\beta\gamma = -\frac{d}{a}$

Relations

$$(x-h)^2 + (y-k)^2 = r^2$$

Financial Mathematics

$$A = P(1+r)^n$$

Sequences and series

$$T_n = a + (n-1)d$$

$$S_n = \frac{n}{2}[2a + (n-1)d] = \frac{n}{2}(a+l)$$

$$T_n = ar^{n-1}$$

$$S_n = \frac{a(1-r^n)}{1-r} = \frac{a(r^n-1)}{r-1},\ r \neq 1$$

$$S = \frac{a}{1-r},\ |r| < 1$$

Logarithmic and Exponential Functions

$$\log_a a^x = x = a^{\log_a x}$$

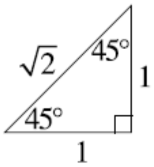
$$\log_a x = \frac{\log_b x}{\log_b a}$$

$$a^x = e^{x \ln a}$$

Trigonometric Functions

$$\sin A = \frac{\text{opp}}{\text{hyp}}, \quad \cos A = \frac{\text{adj}}{\text{hyp}}, \quad \tan A = \frac{\text{opp}}{\text{adj}}$$

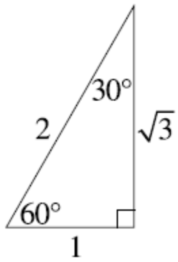
$$A = \frac{1}{2}ab\sin C$$



$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$c^2 = a^2 + b^2 - 2ab\cos C$$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$



$$l = r\theta$$

$$A = \frac{1}{2}r^2\theta$$

Trigonometric identities

$$\sec A = \frac{1}{\cos A}, \ \cos A \neq 0$$

$$\operatorname{cosec} A = \frac{1}{\sin A}, \ \sin A \neq 0$$

$$\cot A = \frac{\cos A}{\sin A}, \ \sin A \neq 0$$

$$\cos^2 x + \sin^2 x = 1$$

Compound angles

$$\sin(A + B) = \sin A\cos B + \cos A\sin B$$

$$\cos(A + B) = \cos A\cos B - \sin A\sin B$$

$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A\tan B}$$

If $t = \tan\frac{A}{2}$ then $\sin A = \frac{2t}{1 + t^2}$

$$\cos A = \frac{1 - t^2}{1 + t^2}$$

$$\tan A = \frac{2t}{1 - t^2}$$

$$\cos A\cos B = \frac{1}{2}\left[\cos(A - B) + \cos(A + B)\right]$$

$$\sin A\sin B = \frac{1}{2}\left[\cos(A - B) - \cos(A + B)\right]$$

$$\sin A\cos B = \frac{1}{2}\left[\sin(A + B) + \sin(A - B)\right]$$

$$\cos A\sin B = \frac{1}{2}\left[\sin(A + B) - \sin(A - B)\right]$$

$$\sin^2 nx = \frac{1}{2}(1 - \cos 2nx)$$

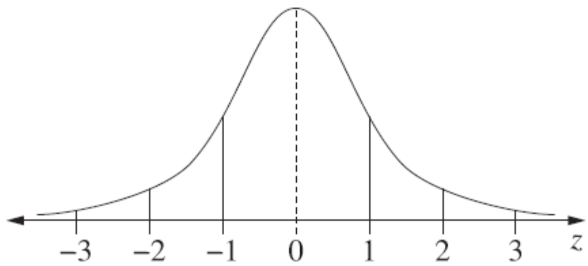
$$\cos^2 nx = \frac{1}{2}(1 + \cos 2nx)$$

Statistical Analysis

$$z = \frac{x - \mu}{\sigma}$$

An outlier is a score
less than $Q_1 - 1.5 \times IQR$
or
more than $Q_3 + 1.5 \times IQR$

Normal distribution



- approximately 68% of scores have z-scores between –1 and 1
- approximately 95% of scores have z-scores between –2 and 2
- approximately 99.7% of scores have z-scores between –3 and 3

$$E(X) = \mu$$

$$\operatorname{Var}(X) = E\left[(X - \mu)^2\right] = E\left(X^2\right) - \mu^2$$

Probability

$$P(A \cap B) = P(A)P(B)$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$P(A|B) = \frac{P(A \cap B)}{P(B)}, \ P(B) \neq 0$$

Continuous random variables

$$P(X \leq x) = \int_a^x f(x)\,dx$$

$$P(a < X < b) = \int_a^b f(x)\,dx$$

Binomial distribution

$$P(X = r) = {}^nC_r p^r(1 - p)^{n-r}$$

$$X \sim \operatorname{Bin}(n, p)$$

$$\Rightarrow \ P(X = x)$$

$$= \binom{n}{x}p^x(1 - p)^{n-x}, \ x = 0, 1, \ldots, n$$

$$E(X) = np$$

$$\operatorname{Var}(X) = np(1 - p)$$

Differential Calculus

Function	Derivative
$y = f(x)^n$	$\frac{dy}{dx} = nf'(x)[f(x)]^{n-1}$
$y = uv$	$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$
$y = g(u)$ where $u = f(x)$	$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$
$y = \frac{u}{v}$	$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$
$y = \sin f(x)$	$\frac{dy}{dx} = f'(x)\cos f(x)$
$y = \cos f(x)$	$\frac{dy}{dx} = -f'(x)\sin f(x)$
$y = \tan f(x)$	$\frac{dy}{dx} = f'(x)\sec^2 f(x)$
$y = e^{f(x)}$	$\frac{dy}{dx} = f'(x)e^{f(x)}$
$y = \ln f(x)$	$\frac{dy}{dx} = \frac{f'(x)}{f(x)}$
$y = a^{f(x)}$	$\frac{dy}{dx} = (\ln a)f'(x)a^{f(x)}$
$y = \log_a f(x)$	$\frac{dy}{dx} = \frac{f'(x)}{(\ln a)f(x)}$
$y = \sin^{-1} f(x)$	$\frac{dy}{dx} = \frac{f'(x)}{\sqrt{1-[f(x)]^2}}$
$y = \cos^{-1} f(x)$	$\frac{dy}{dx} = -\frac{f'(x)}{\sqrt{1-[f(x)]^2}}$
$y = \tan^{-1} f(x)$	$\frac{dy}{dx} = \frac{f'(x)}{1+[f(x)]^2}$

Integral Calculus

$$\int f'(x)[f(x)]^n dx = \frac{1}{n+1}[f(x)]^{n+1} + c$$

where $n \neq -1$

$$\int f'(x)\sin f(x)dx = -\cos f(x) + c$$

$$\int f'(x)\cos f(x)dx = \sin f(x) + c$$

$$\int f'(x)\sec^2 f(x)dx = \tan f(x) + c$$

$$\int f'(x)e^{f(x)}dx = e^{f(x)} + c$$

$$\int \frac{f'(x)}{f(x)}dx = \ln|f(x)| + c$$

$$\int f'(x)a^{f(x)}dx = \frac{a^{f(x)}}{\ln a} + c$$

$$\int \frac{f'(x)}{\sqrt{a^2-[f(x)]^2}}dx = \sin^{-1}\frac{f(x)}{a} + c$$

$$\int \frac{f'(x)}{a^2+[f(x)]^2}dx = \frac{1}{a}\tan^{-1}\frac{f(x)}{a} + c$$

$$\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$$

$$\int_a^b f(x)dx$$

$$\approx \frac{b-a}{2n}\left\{f(a) + f(b) + 2\left[f(x_1) + \cdots + f(x_{n-1})\right]\right\}$$

where $a = x_0$ and $b = x_n$

2021 Trial Higher School Certificate Examination
Mathematics Advanced

Student Number ____________________ Teacher ____________________

Section I – Multiple Choice Answer Sheet

Allow about 25 minutes for this section

Select the alternative A, B, C or D that best answers the question. Fill in the response oval completely.

Sample: $2 + 4 =$ (A) 2 (B) 6 (C) 8 (D) 9

A ○ B ● C ○ D ○

If you think you have made a mistake, put a cross through the incorrect answer and fill in the new answer.

A ● B ⊗ C ○ D ○

If you change your mind and have crossed out what you consider to be the correct answer, then indicate the correct answer by writing the word **correct** and drawing an arrow as follows.

correct

A ⊗ B ⊗ C ○ D ○

1. A ○ B ○ C ○ D ○
2. A ○ B ○ C ○ D ○
3. A ○ B ○ C ○ D ○
4. A ○ B ○ C ○ D ○
5. A ○ B ○ C ○ D ○
6. A ○ B ○ C ○ D ○
7. A ○ B ○ C ○ D ○
8. A ○ B ○ C ○ D ○
9. A ○ B ○ C ○ D ○
10. A ○ B ○ C ○ D ○

Section I

10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1 – 10

1. The amplitude of the curve $y = \frac{1}{2}\sin(3x + \frac{\pi}{6})$ is **MA12.5**

A. $\frac{1}{6}$

B. $\frac{\pi}{6}$

C. $\frac{1}{3}$

D. $\frac{1}{2}$

2. What is the equation of the tangent to the curve $y = \sin x$ at the origin? **MA12.6**

A. $y = -x$

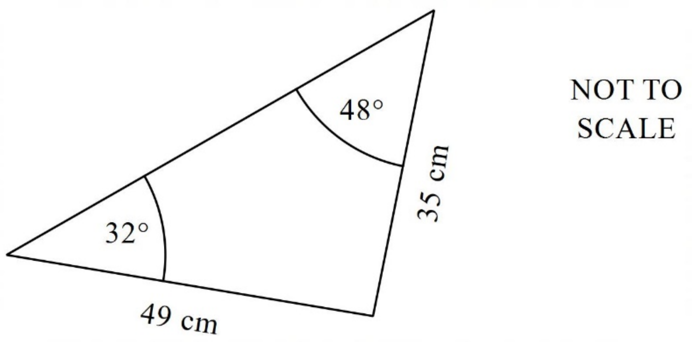
B. $y = \cos x$

C. $y = \sin x$

D. $y = x$

3. Calculate the area of the triangle below. **MA11.4**

(Answer to the nearest cm^2.)



A. $422\ cm^2$

B. $637\ cm^2$

C. $844\ cm^2$

D. $858\ cm^2$

4. Given that $\log_a b = 3.75$ and $\log_a c = 0.25$. **MA11.6**

Find $\log_a (bc)^2$.

A. 0.9375

B. 3.5

C. 8

D. 0.879

5. Find the exact value of $\int_0^4 \sqrt{16 - x^2}\, dx$. **MA12.7**

A. 4 $units^2$

B. 16π $units^2$

C. 2 $units^2$

D. 4π $units^2$

6. The point $P(8, -3)$ lies on the graph of $y = f(x)$. **MA12.1**

Find the coordinates of the image of P if the function is transformed to :

$y = -2f(x + 7) + 5$.

A. (1, 11)

B. (1, –1)

C. (15, 11)

D. (15, –1)

7. Which interval gives the range of the function **MA12.1**

$y = 2\sqrt{25 - x^2}$?

A. [- 5 , 5]

B. [0 , 10]

C. [-10 , 10]

D. [0 , 5]

8. Simplify $\sec^2\theta - \tan^2\theta$. **MA11.4**

A. 1

B. $\sin^2\theta$

C. $\cot^2\theta$

D. 2

9. The third term of an arithmetic series is 32 and the sixth term is 17. **MA12.4**

Find the sum of the first ten terms.

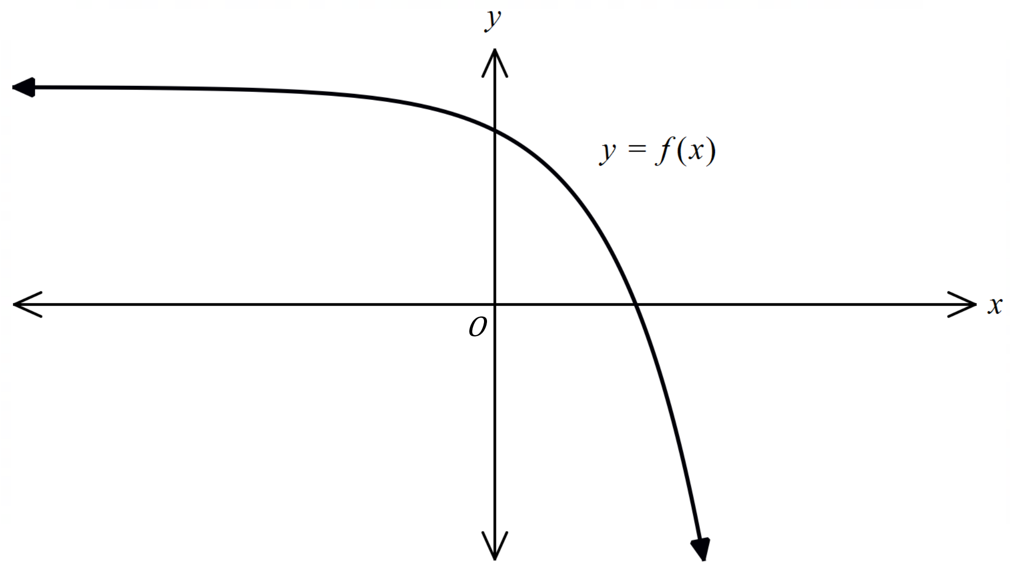
A. - 5

B. 42

C. 49

D. 195

10. The graph of the relation $y = f(x)$ is shown below: **MA12.3**



Which statement is correct?

A. $f\,'(x) > 0$ and $f\,''(x) < 0$

B. $f\,'(x) < 0$ and $f\,''(x) < 0$

C. $f\,'(x) > 0$ and $f\,''(x) > 0$

D. $f\,'(x) < 0$ and $f\,''(x) > 0$

END OF SECTION I

Section II: Questions 11-33

Instructions • Answer the questions in the spaces provided. Sufficient spaces are provided for typical responses.

- Your responses should include relevant mathematical reasoning and/or calculations.
- Extra writing space is provided at the back of the booklet. If you use this space, clearly indicate which question you are answering.

Question 11 (7 marks)

(a) Express n in terms of x and y, given $(\sqrt{2})^n = \frac{4^x}{32^y}$ **1**

MA12.1

$2^{\frac{n}{2}} = \frac{2^{2x}}{2^{5y}}$

$2^{\frac{n}{2}} = 2^{2x-5y}$

$\frac{n}{2} = 2x - 5y$

$n = 4x - 10y$

(b) One of the two solutions to the equation below is $x = 1$ **2**

$2^{2x+y} - 2^{x+y} = 9(2^x) - 2$. Find the value of y. **MA12.1**

$2^{2+y} - 2^{1+y} = 9(2) - 2$

$2^{2+y} - 2^{1+y} = 16$

$2^2(2^y) - 2(2^y) = 16$

$2^y(4-2) = 16$

$2^y = 8$ $\quad 2^y = 2^3 \quad \therefore y = 3$

(c) What is the other value of x for the equation. **2**

MA12.1

$2^{2x+3} - 2^{x+3} = 9(2^x) - 2$

$2^3(2^x)^2 - 2^3(2^x) = 9(2^x) - 2$

$8(2^x)^2 - 8(2^x) - 9(2^x) + 2 = 0$

$8(2^x)^2 - 17(2^x) + 2 = 0$

$2^x = m$

$8m^2 - 17m + 2 = 0$

$8m^2 - 16m - m + 2 = 0$

$8m(m-2) - 1(m-2) = 0$

$(m-2)(8m-1) = 0$

$m = 2$ $\quad m = \frac{1}{8}$

$2^x = 2 \quad \therefore x = 1$ $\quad 2^x = 2^{-3}$ $\quad x = -3$

(d) Find the values of a and b such that

$$\sum_{n=1}^{3} \log_2 2n = a + \log_2 b$$ **2**

MA12.4

LHS: $\log_2 2 + \log_2 4 + \log_2 6$

$= \log_2 2 + 2\log_2 2 + \log_2 6$

$= 3\log_2 2 + \log_2 6$

$= 3 + \log_2 6$

$a = 3 \qquad b = 6$

Question 12 (2 marks) **MA12.1**

The damage caused by a moving car when it hits an object is called the 'collision impact' and is proportional to the square of the speed of the car.

i) Write and equation to describe the relationship between collision impact (C) and the speed of the car (V) for some constant (K). **1**

$C \propto V^2$

$C = KV^2$

ii) What happens to the collision impact when the speed of a car is reduced to one third? **1**

$C = K\left(\frac{V}{3}\right)^2$

$= K\frac{V^2}{9}$

$C = \frac{1}{9}(KV^2)$

∴ The Collision impact will be $\frac{1}{9}$ of the original impact.

Question 13 (4 marks)

Desmond tosses a pair of dice. The sum on the uppermost faces are recorded.

(a) i. What is the probability that the sum is 9? **1**

MA11.7

(3, 6), (6,3), (5,4)
(4,5)

$\therefore P(\text{Sum of } 9) = \frac{4}{36} = \frac{1}{9}$

ii. Find the probability that he will throw a sum of 9 before he throws a sum of 7. **2**

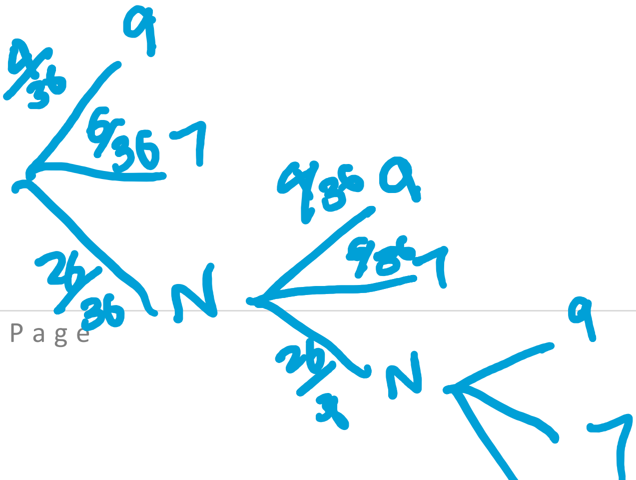
MA11.7

P(Sum of 9 before sum of 7)

$= \frac{1}{9} + \frac{26}{36} \times \frac{1}{9} + \left(\frac{26}{36}\right)^2 \times \frac{1}{9}$
$+ \text{----}$

$S_\infty = \frac{a}{1-r} = \frac{\frac{1}{9}}{1 - \frac{13}{18}} = \frac{2}{5}$

Sum of 7
(1,6) (6,1)
(2,5), (5,2)
(3,4), (4,3)



(b)

If X is a random variable such that $P(X \geq 3) = m$ and $P(X \leq 9) = n$. Show that $P(X < 3|X > 9)$ is $\frac{m-1}{n-1}$ **1**

MA11.7

$$P(x<3|x<9) = \frac{P(x<3 \cap x<9)}{P(x<9)}$$

$$= \frac{P(x<3)}{P(x>9)}$$

$$= \frac{1-m}{1-n}$$

$$= \frac{-(m-1)}{-(n-1)} = \frac{m-1}{n-1}$$

Question 14 (4 marks)

Differentiate the following with respect to x:

(a) $\dfrac{\cos 3x}{x^2}$. **2**

MA12.6

$u = \cos 3x \qquad v = x^2$

$u' = -3\sin 3x \qquad v' = 2x$

$y' = \dfrac{-3x^2 \sin 3x - 2x\cos 3x}{x^4}$

$= \dfrac{-x(3x\sin 3x + 2\cos 3x)}{x^3}$

$= -\dfrac{(3x\sin 3x + 2\cos 3x)}{x^3}$

(b) $\ln\sqrt{3-x}$. $= \ln(3-x)^{\frac{1}{2}} = \frac{1}{2}\ln(3-x)$ **2**

MA12.6

$y' = \frac{1}{2}\left(\dfrac{-1}{3-x}\right)$

$= \dfrac{-1}{2(3-x)} = \dfrac{1}{2(x-3)}$

Question 15 (3 marks)

(a) Find $\int (3x - 4)^8 \, dx$. **1**

MA12.7

$= \dfrac{(3x-4)^9}{9 \times 3} + C$

$= \dfrac{(3x-4)^9}{27} + C$

(b) Find $\int \dfrac{4 \sin\left(\frac{5x}{3}\right)}{7} \, dx$. **2**

MA12.7

$\frac{3}{5} \times \frac{4}{7} \int \frac{5}{3} \times \sin\left(\frac{5x}{3}\right)$

$= \frac{12}{35}\left[-\cos\left(\frac{5x}{3}\right)\right] + C$

$= -\frac{12}{35}\cos\left(\frac{5x}{3}\right) + C$

Question 16(4 marks) **MA11.3**

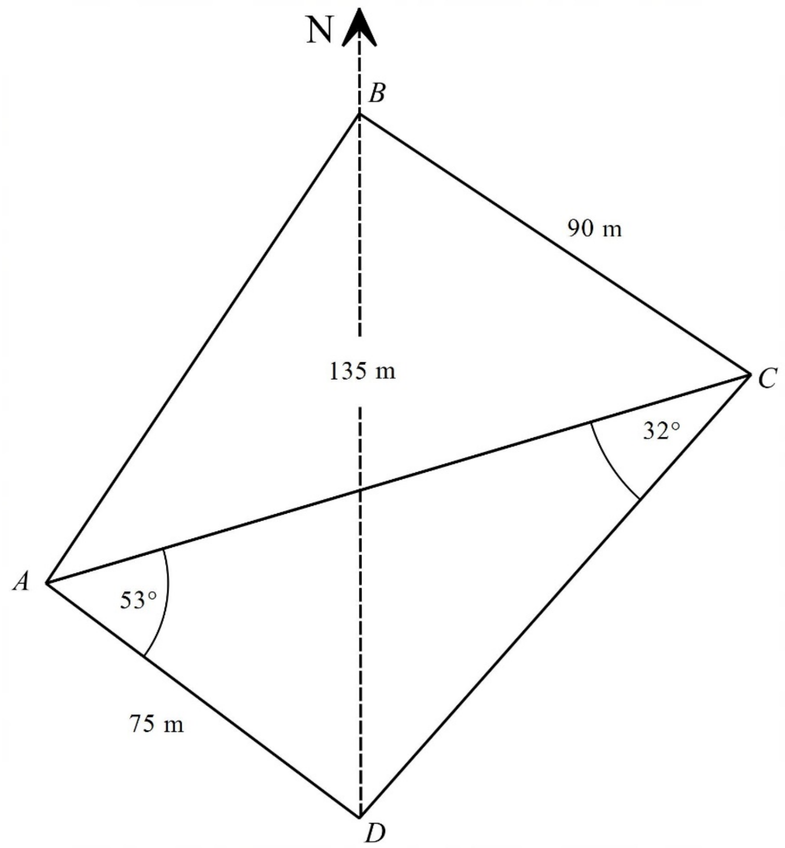
A block of land is in the shape of the quadrilateral *ABCD* shown below. The diagonal *AC* has been drawn.

The following measurements have been taken:

AD = 75 m, *BC* = 90 m.

D is 135 m due south of *B*.

$\angle CAD = 53°$ and $\angle ACD = 32°$.



(a) Find the length of *CD*. (Answer correct to the nearest metre.) **2**

$\frac{CD}{\sin 53} = \frac{75}{\sin 32}$

$CD = \frac{75 \sin 53}{\sin 32} = 113 \text{ m}$

(b) Find the size of $\angle CBD$ and hence give the bearing of *C* from *B*. (Give your answer correct to the nearest degree.) **2**

$\cos(\angle CBD) = \frac{90^2 + 135^2 - 113^2}{2(90)(135)}$

$\angle CBD = 56°$ ∴ Bearing $= 180 - 56$ $= 124° T$

Question 17 (7 marks)

(a) Use the differentiation by first principles to verify that if, the gradient function of $y = 3x^2 - x$ is $y' = 6x - 1$. **2**

MA12.3

$f(x) = 3x^2 - x$

$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$= \lim_{h \to 0} \frac{3(x+h)^2 - (x+h) - (3x^2 - x)}{h}$

$= \lim_{h \to 0} \frac{3(x^2 + 2xh + h^2) - x - h - 3x^2 + x}{h}$

$= \lim_{h \to 0} \frac{3x^2 + 6xh + 3h^2 - x - h - 3x^2 + x}{h}$

$= \lim_{h \to 0} \frac{h(6x + 3h - 1)}{h}$

$= \lim_{h \to 0} 6x + 3h - 1$ at $h = 0$

$f'(x) = 6x - 1$

(b) The line $y = mx + c$ is a normal to the curve $y = x^3 - 3x + 1$ at the point (-2,-1). **2**

Find the values of m and c. **MA12.3**

$m_{tang} = y' = 3x^2 - 3$ at $(-2, -1)$

$m_{tang} = 3(-2)^2 - 3 = 9$

$m_{norm} = \frac{-1}{9}$

$y + 1 = \frac{-1}{9}(x + 2)$

$y = \frac{-1}{9}x - \frac{2}{9} - 1$

$y = \frac{-1}{9}x - \frac{11}{9}$

$\therefore m = \frac{-1}{9}$

$c = \frac{-11}{9}$

(c) If $a = e^x$, simplify $\log_e a^2$ **1**

MA11.6

$\log_e a^2 = 2\log_e a$

$= 2\log_e e^x$

$= 2x \log_e e$

$= 2x$

(d) Solve: **2**

MA11.6

$$\log_e x - \frac{3}{\log_e x} = 2$$

let $m = \log_e x$

$m - \frac{3}{m} = 2$

$m^2 - 3 = 2m$ $\quad m^2 - 2m - 3 = 0$

$(m-3)(m+1) = 0$

$m = 3$ $\quad m = -1$

$\log_e x = 3$ $\quad \log_e x = -1$

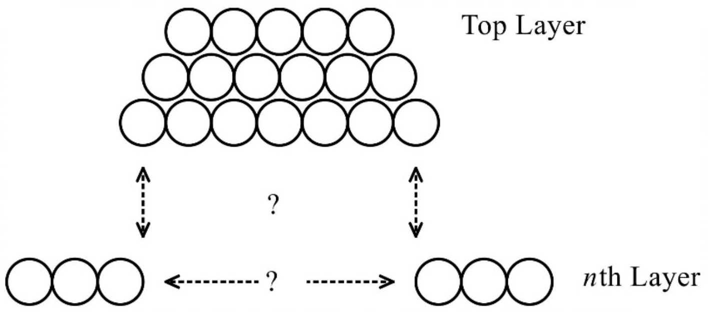
$x = e^3$ $\quad e^{-1} = x$

$x = e^3$ $\quad x = \frac{1}{e}$

Question 18 (4 marks) **MA12.4**

Lachlan works in a Moolworths store and he is making a stack of oranges against a sloping display panel.

The oranges are stacked in layers, as shown, where each layer contains one orange less than the layer below it.



When he has finished, there are five oranges in the top layer, six in the next and so on.

There are n layers altogether.

(a) Show there are $\frac{1}{2}n(n+9)$ oranges in the stack. **2**

$a = 5 \quad d = 1$

$S_n = \frac{n}{2}\left[2(5) + (n-1)\times 1\right]$

$= \frac{n}{2}\left[10 + n - 1\right]$

$= \frac{n}{2}(n+9) = \frac{1}{2}n(n+9)$

(b) If Lachlan has 300 oranges to create his display, how many full rows can he create, if the top row still contains five oranges? **2**

$300 = \frac{1}{2}n(n+9)$

$600 = n^2 + 9n$

$0 = n^2 + 9n - 600$

$n = \frac{-9 \pm \sqrt{9^2 - 4(1)(-600)}}{2} = 20.4$ or -29.4

$\therefore$ 20 Rows can be created.

Question 19 (3 marks) **MA11.7**

The random variable $\boldsymbol{X}$ has this probability distribution.

X	0	1	2	3	4
$\Pr(X = x)$	0.1	0.2	0.4	0.2	0.1

Find

i. $P(X > 1|X \leq 3)$ **1**

$P(X>1|\leq 3) = \dfrac{P(X>1 \cap X\leq 3)}{P(X\leq 3)}$

$= \dfrac{0.4+0.2}{0.9} = \dfrac{2}{3}$

ii. $P(X)$, the variance of X **2**

$\mu = 0(0.1) + 1(0.2) + 2(0.4) + 3(0.2) + 4(0.1)$

$= 2$

$Var(X) = 0^2(0.1) + 1^2(0.2) + 2^2(0.4) + 3^2(0.2) + 4^2(0.1) - 2^2$

$= 1.2$

Question 20 (3 marks) **MA11.4**

1

(a) Express $5cot^2x - 2cosecx + 2$ in terms of $cosecx$

$1 + Cot^2x = Cosec^2x$

$5(Cosec^2x - 1) - 2Cosecx + 2$

$= 5Cosec^2x - 5 - 2Cosecx + 2$

$= 5Cosec^2x - 2Cosecx - 3$

2

(b) Hence, solve the equation $5cot^2x - 2cosecx + 2 = 0$ for $0 \leq x \leq 2\pi$.

$5Cosec^2x - 2Cosecx - 3 = 0$

$5cosec^2x - 5Cosecx + 3Cosecx - 3 = 0$

$5Cosecx(Cosecx - 1) + 3(Cosecx - 1) = 0$

$(Cosecx - 1)(5Cosecx + 3) = 0$

$Cosecx = 1$ $\quad$ $Cosecx = -\frac{3}{5}$

$x = \frac{\pi}{2}$ $\quad$ No solution

$\therefore x = \frac{\pi}{2}$ only solution

Question 21 (3 marks)

(a) Find $\frac{d}{dx}(\log_e(cosx))$ **1**

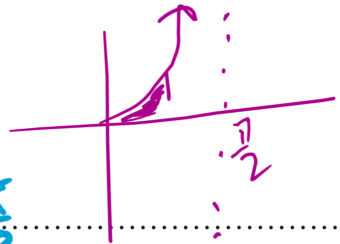
MA12.7

$= \frac{-\sin x}{\cos x}$

$= -\tan x$

(b) Hence, show that the area enclosed by the curve $y = tanx$, the x- axis and the ordinate $x = \frac{\pi}{3}$ is $\boldsymbol{ln2}$. **2**

MA12.7



$A = -\int_0^{\frac{\pi}{3}} -\tan x \, dx$

$= -\left[\log_e(\cos x)\right]_0^{\frac{\pi}{3}}$

$= -\left[\log_e \frac{1}{2} - \log_e 1\right]$

$= -\log_e 2^{-1} = \log_e 2 = \ln 2$

Question 22 (4 marks)

An infinite geometric series is given by $1 - tan^2x + tan^4x - tan^6x + \cdots$. for $0 \leq x \leq 2\pi$

(a) Find the limiting sum of the series in simplest form **2** **MA12.4**

$r = -\tan^2 x$ $\quad -1 < -\tan^2 x < 1$

$-1 < \tan^2 x < 1$

$S_\infty = \frac{a}{1-r}$ $\quad -1 < \tan x < 1$

$= \frac{1}{1-(-\tan^2 x)} = \frac{1}{1+\tan^2 x} = \frac{1}{\sec^2 x}$

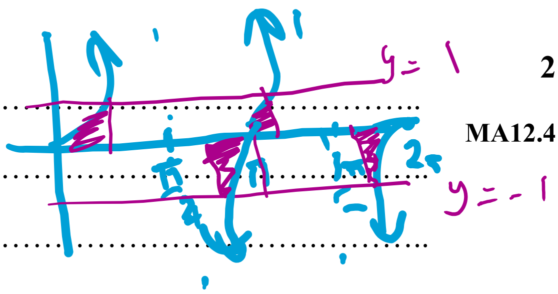
$= \cos^2 x$

(b) For what values of x is the limiting sum is equal $\frac{1}{4}$? **2** **MA12.4**

$\frac{1}{4} = \cos^2 x$

$\cos x = \pm \frac{1}{2}$

$x = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$



Solution must lie

$0 \leq x < \frac{\pi}{4}$

$\frac{3\pi}{4} < x \leq \pi$

$\pi \leq x < \frac{5\pi}{4}$

$\frac{7\pi}{4} < x \leq 2\pi$

Since all the Solution above don't lie within the permitted values of r ∴ No solution

Question 23 (3 marks)

(a) Complete the table of values for $y = \sqrt{1 - x^2}$. **1**

Answer to 3 significant figures where required. **MA12.7**

x	0	0.125	0.25	0.375	0.5
y	1	0.992	0.968	0.927	0.866

(b) By using the Trapezoidal rule with 4 sub intervals, estimate the integral **2**

$$\int_0^{\frac{1}{2}} \sqrt{1 - x^2}\, dx \,.$$

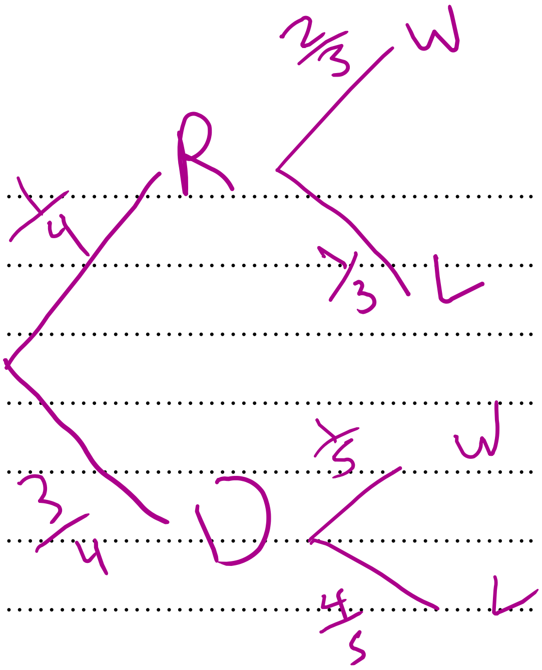
$\doteqdot \frac{0.5 - 0}{2(4)}\left[1 + 0.866 + 2(0.992 + 0.968 + 0.927)\right]$

$\doteqdot 0.4775$

Question 24 (3 marks) **MA11.7**

The manager of a team notices that the team has a probability of $\frac{2}{3}$ of winning the game if it is raining and if it is dry, the probability of the team winning is $\frac{1}{5}$. The probability that it will rain on a day when they play is $\frac{1}{4}$.

(a) Find the probability that they will not win. **1**



P(not winning)

$= \frac{1}{4} \times \frac{1}{3} + \frac{3}{4} \times \frac{4}{5}$

$= \frac{1}{12} + \frac{3}{5}$

$= \frac{41}{60}$

(b) Given that the team has won the game, calculate the probability that it rained on the day of the match. **2**

$P(R \mid W) = \dfrac{\frac{1}{4} \times \frac{2}{3}}{\frac{1}{4} \times \frac{2}{3} + \frac{3}{4} \times \frac{1}{5}}$

$= \frac{10}{19}$

Question 25 (7 marks)

Consider the curve $y = \frac{1}{4}x^4 - x^3$.

(a) Find any stationary points and determine their nature. **3**

$y' = x^3 - 3x^2$

$0 = x^2(x-3)$

$x = 0 \qquad x = 3$

$y = 0 \qquad y = -\frac{27}{4}$

$\therefore$ SP $(0,0)$ and $\left(3, -\frac{27}{4}\right)$

$y'' = 3x^2 - 6x$

at $(0,0)$ $f''(0) = 0$ possible pt of inflexion

x	-0.1	0	0.1
y''	0.63	0	-0.57

Since concavity changes $\therefore$ $(0,0)$ is a horizontal pt of inflexion.

at $\left(3, -\frac{27}{4}\right)$

$f''(3) = 3(3)^2 - 6(3)$

$= 9 > 0$ ∪ min pt

$\therefore$ $(0,0)$ horizontal pt of inflexion

$\left(3, -\frac{27}{4}\right)$ is minimum pt

(b) Find any points of inflection. **1**

$y'' = 3x^2 - 6x$

$0 = 3x(x-2)$

$x = 0$, $y = 0$ $\qquad$ $x = 2$, $y = -4$

x	1.9	2	2.1
y''	-0.57	0	0.63

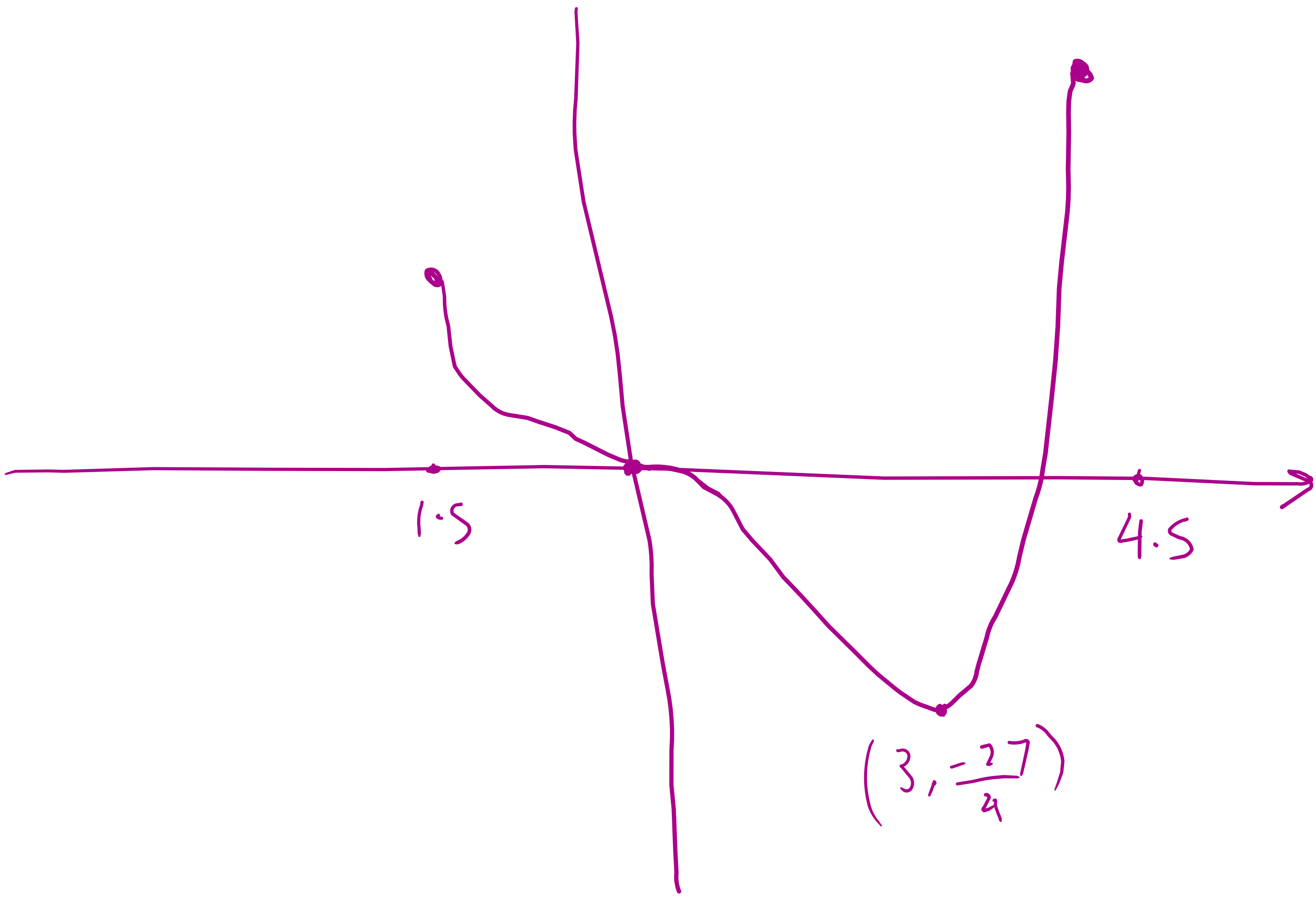
Since concavity changes ∴ $(2, -4)$ is a pt of inflexion

2

(c)

Sketch the curve for $-1.5 \leq x \leq 4.5$, showing all x- intercepts.

$x = -1.5$, $y = 4.64$ $\qquad$ $x = 4.5$, $y = 11.39$



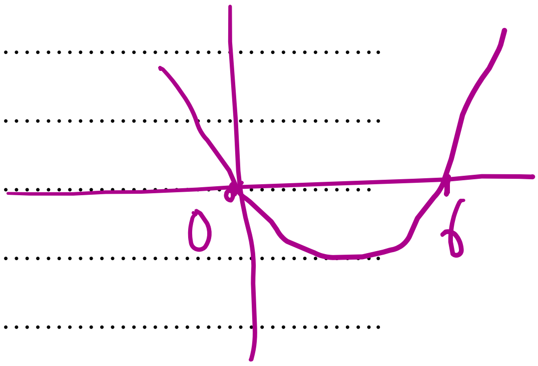
(d) For what values of x is the curve concave down? **1**

$y'' < 0$

$3x^2 - 6x < 0$

$3x(x-6) < 0$

$0 < x < 6$

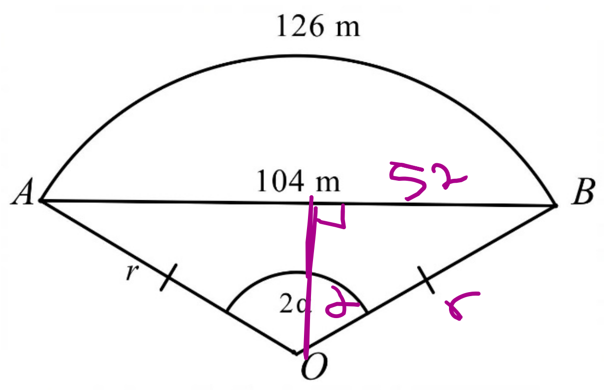


Question 26 (4 marks)

A sector of a circle, centre O, is shown below.

The points A and B lie on the circle, such that the length of the chord AB is 104 metres, and the length of the arc AB

The radius is r metres and the angle subtended at the centre by the arc is 2α radians.



(a) Show that $\sin\alpha = \frac{52\alpha}{63}$.

$l = r\theta$

$126 = r\theta$ $\qquad r = \frac{126}{2\alpha} = \frac{63}{\alpha}$

$\sin\alpha = \frac{52}{r}$

$\sin\alpha = \frac{52}{\frac{63}{\alpha}} = \frac{52\alpha}{63}$

(b) If $\alpha = \frac{\pi}{3}$, find the exact radius of the circle.

Express your answer with a rational denominator.

$\sin\alpha = \frac{52}{r}$

$r = \frac{52}{\sin\frac{\pi}{3}} = \frac{52}{\frac{\sqrt{3}}{2}}$

$r = \frac{104}{\sqrt{3}} = \frac{104\sqrt{3}}{3}$ m

Question 27 (3 marks) **MA12.3**

If $y = \frac{\log_e x}{x}$

(a) Find $\frac{dy}{dx}$ **1**

$u = \log_e x \qquad v = x$

$u' = \frac{1}{x} \qquad v' = 1$

$y' = \frac{\frac{1}{x}(x) - \log_e x}{x^2} = \frac{1 - \log_e x}{x^2}$

(b) Hence show that $\int_e^{e^2} \frac{1-\log_e x}{x\log_e x}\,dx = \log_e 2 - 1$ **2**

$$\frac{1-\log_e x}{x\log_e x} = \frac{\frac{1-\log_e x}{x^2}}{\frac{\log_e x}{x}}$$

$$\int_e^{e^2} \frac{\left(\frac{1-\log_e x}{x^2}\right) f'(x)}{\left(\frac{\log_e x}{x}\right) f(x)} = \ln\left|\left[\frac{\log_e x}{x}\right]_e^{e^2}\right|$$

$$= \ln\left[\frac{\log_e e^2}{e^2}\right] - \ln\left[\frac{\log_e e}{e}\right]$$

$$= \ln\left(\frac{2}{e^2}\right) - \ln\frac{1}{e}$$

$$= \ln 2 - \ln e^2 + \ln e$$

$$= \ln 2 - 2 + 1 = \ln 2 - 1$$

$$= \log_e 2 - 1$$

Question 28 (4 marks)

The length of daylight, *L(t)*, is defined as the number of hours from sunrise to sunset, and can be modelled by the equation $L(t) = 12 + \cos\left(\frac{2\pi t}{366}\right)$ where *t* is the number of days after 21 December 2015, for $0 \leq t \leq 366$. **MA12.5**

(a) Find the length of daylight on 21 December 2015. **1**

$t = 0$

$L(t) = 12 + \cos(0) = 13$ hours

(b) What is the shortest length of daylight? **1**

amp = 1 Centre = 12

∴ shortest day length = 12 − 1
= 11 hours

(c) What are the two values of *t* for which the length of daylight is 12? **2**

$12 = 12 + \cos\left(\frac{2\pi t}{366}\right)$

$0 = \cos\left(\frac{2\pi t}{366}\right)$

$\frac{2\pi t}{366} = \frac{\pi}{2}, \frac{3\pi}{2}$

$t = \left(\frac{\pi}{2}\right) \div \frac{2\pi}{366}, \frac{3\pi}{2} \div \frac{2\pi}{366}$

$= 91.5, 279.5$

Question 29 (3 marks)

Given that the function $f(x)$ has a derivative $y' = 4e^{4x} + 3$ and the equation of the tangent to this curve is $y = 7x + 2$. Find the exact value of $f(3)$. **3**

MA12.6

$m_{tan} = 7$

$7 = 4e^{4x} + 3$

$4 = 4e^{4x}$

$1 = e^{4x}$

$e^0 = e^{4x}$

$x = 0$

$y = 7(0) + 2$

$(0, 2)$

$y = \int 4e^{4x} + 3 \, dx$

$y = e^{4x} + 3x + C$

$2 = e^0 + 3(0) + C$

$C = 1$

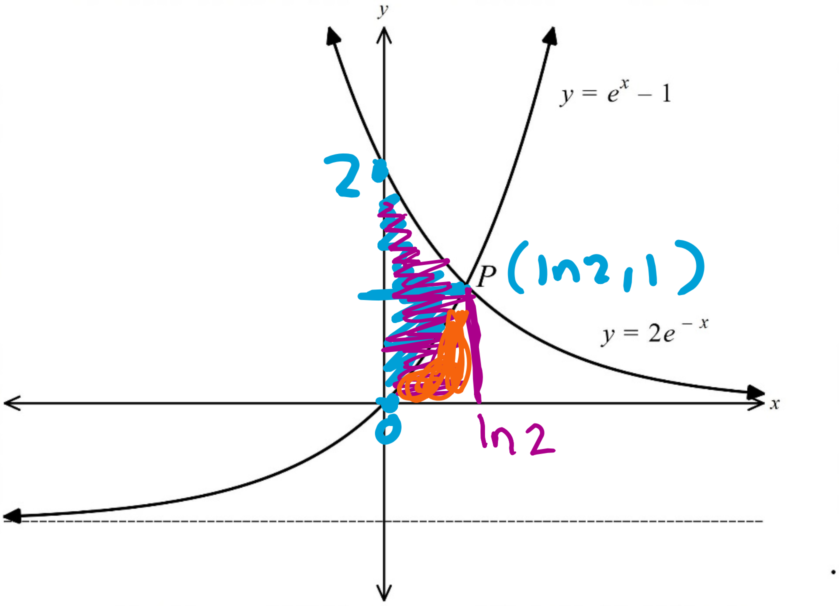
$y = e^{4x} + 3x + 1$

$f(3) = e^{12} + 12 + 1$

$f(3) = e^{12} + 13$

Question 30 (4 marks)

Two curves $y = 2e^{-x}$ and $y = e^{x} - 1$ intersect at a point P.



(a) Show, algebraically, that the coordinates of P are $(\ln 2,\ 1)$. **2**

$y = 2e^{-x}$ $\quad$ $y = e^{x} - 1$ $\quad$ Solve simultaneously

$e^{x} - 1 = \frac{2}{e^{x}}$

$e^{2x} - e^{x} = 2$

$(e^{x})^2 - e^{x} - 2 = 0$

$(e^{x} - 2)(e^{x} + 1) = 0$

$e^{x} = 2$ $\qquad$ $e^{x} = -1$ No solution

$x = \ln 2$

$y = e^{\ln 2} - 1$

$= 2 - 1 = 1$

$\therefore P(\ln 2, 1)$

(b) Find the area bounded by the two curves and the y-axis. **2**

$$A = \int_0^{\ln 2} 2e^{-x} - e^{x} + 1 \, dx$$

$$= \left[-2e^{-x} - e^{x} + x \right]_0^{\ln 2}$$

$$= \left[-2e^{-\ln 2} - e^{\ln 2} + \ln 2 \right] - \left[-2e^{0} - e^{0} + 0 \right]$$

$$= -2\left(\tfrac{1}{2}\right) - 2 + \ln 2 + 2 + 1$$

$$= \ln 2$$

Question 31 (2 marks) **MA12.4**

Alex buys a tractor under a buy-back scheme. This scheme gives Alex the right to sell the tractor back to the dealer.

The recurrence relation below can be used to calculate the price Alex sells the tractor back to the dealer (P_n), after n years.

$$P_{0=56000}, \quad P_{n=} P_{n-1} - 7000$$

(a) Write the general rule to find the value of p_n in terms of n. **1**

$P_1 = 56000 - 7000$

$P_2 = P_1 - 7000$

$P_3 = P_2 - 7000$

$P_n = 56000 - 7000n$

(b) After how many years will the dealer offer to buy back Alex's tractor at half of its original value. **1**

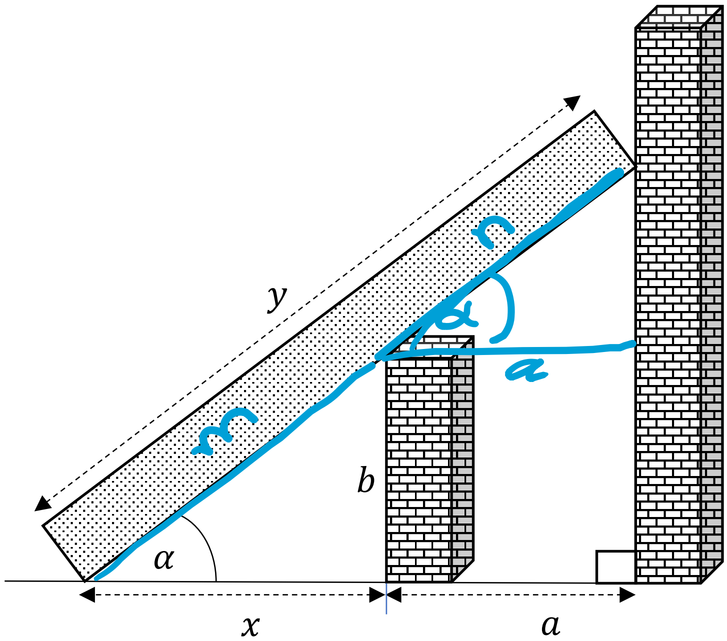
$28000 = 63000 - 7000n$

$7000n = 35000$

$n = 5$ years

Question 32 (6 marks) **MA12.3**

A vertical wall in danger of collapse is to be braced by a beam, which must pass over a second lower wall b metres high and located a metres from the first wall. Let the length of the beam be y metres, the angle the beam makes with the horizontal be α and x is the distance from the foot of the beam to the smaller wall.



2

i. Show that $y = a sec\alpha + b cosec\alpha$

$\sin\alpha = \frac{b}{m}$ $\quad m = \frac{b}{\sin\alpha} = b\,\text{Cosec}\,\alpha$

$\cos\alpha = \frac{a}{n}$ $\quad n = \frac{a}{\cos\alpha} = a\,\text{Sec}\,\alpha$

$y = m + n$

$y = a\,\text{Sec}\,\alpha + b\,\text{Cosec}\,\alpha$

ii. By finding the stationary points on the curve $y = a\sec\alpha + b\,cosec\,\alpha$,

Prove that $\tan\alpha = \sqrt[3]{\frac{b}{a}}$ **2**

$y = (a\cos\alpha)^{-1} + b(\sin\alpha)^{-1}$

$y' = -a(\cos\alpha)^{-2} \times -\sin\alpha - b(\sin\alpha)^{-2} \times \cos\alpha$

$0 = \frac{a\sin\alpha}{\cos^2\alpha} - \frac{b\cos\alpha}{\sin^2\alpha}$

$0 = \frac{a\sin^3\alpha - b\cos^3\alpha}{\cos^2\alpha\sin^2\alpha}$

$0 = a\sin^3\alpha - b\cos^3\alpha$

$b\cos^3\alpha = a\sin^3\alpha$

$\frac{b}{a} = \tan^3\alpha$

$\tan\alpha = \sqrt[3]{\frac{b}{a}}$

iii. Hence show that the shortest beam that can be used is given by **2**

$$y = a\sqrt{1 + \left(\frac{b}{a}\right)^{\frac{2}{3}}} + b\sqrt{1 + \left(\frac{a}{b}\right)^{\frac{2}{3}}}$$

$\tan^3\alpha = \frac{b}{a}$ $\tan\alpha = \frac{\sqrt[3]{b}}{\sqrt[3]{a}}$

$y' = \frac{a\sin\alpha}{\cos^2\alpha} - \frac{b\cos\alpha}{\sin^2\alpha}$

$u = a\sin\alpha$ $u = b\cos\alpha$

$u' = a\cos\alpha$ $u' = -b\sin\alpha$

$v = \cos^2\alpha$ $v = \sin^2\alpha$

$v' = -2\cos\alpha\sin\alpha$ $v' = 2\sin\alpha\cos\alpha$

$y'' = \frac{a\cos^3\alpha + 2a\sin^2\alpha\cos\alpha}{\cos^4\alpha} - \frac{-b\sin^3\alpha - 2b\sin\alpha\cos^2\alpha}{\sin^4\alpha}$

$= \frac{a\cos^3\alpha + 2a\sin^2\alpha\cos\alpha}{\cos^4\alpha} + \frac{b\sin^3\alpha + 2b\sin\alpha\cos^2\alpha}{\sin^4\alpha} > 0$

$\therefore$ min point

$\tan\alpha = \sqrt[3]{\frac{b}{a}}$ $1 + \tan^2\alpha = \sec^2\alpha$

$1 + \left(\frac{b}{a}\right)^{\frac{2}{3}} = \sec^2\alpha$

$\cot\alpha = \sqrt[3]{\frac{a}{b}}$ $1 + \cot^2\alpha = \csc^2\alpha$

$1 + \left(\frac{a}{b}\right)^{2/3} = \csc^2\alpha$

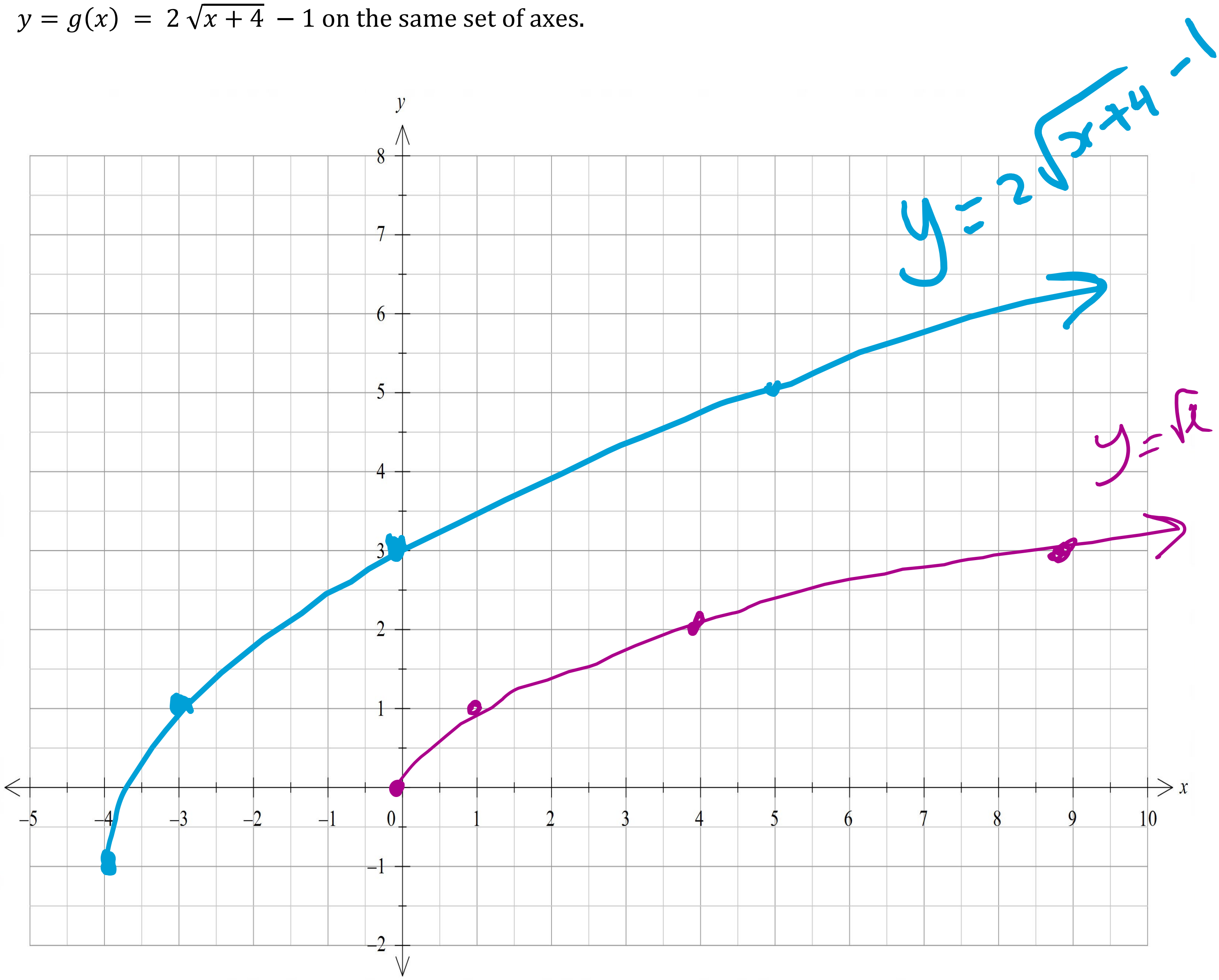
$y = a\sec\alpha + b\csc\alpha$

$y = a\sqrt{1 + \left(\frac{b}{a}\right)^{\frac{2}{3}}} + b\sqrt{1 + \left(\frac{a}{b}\right)^{2/3}}$

Question 33 (3 marks) **MA12.1**

3

On the axes provided below, draw a sketch of $y = f(x) = \sqrt{x}$ and use this to draw a sketch of $y = g(x) = 2\sqrt{x+4} - 1$ on the same set of axes.



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